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THE

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On the true Measure of the Probabilities of Survivorship between two Lives. *By* PETER GRAY, Esq., F.R.A.S.

[Read before the Institute of Actuaries, 27th of January, 1851.]

A GOOD deal of misconception prevails, on the part of many interested in the science of life contingencies, in regard to the expressions usually assigned as the measures of the probabilities of the different orders of survivorship among two or more lives. It seems to be supposed that these expressions are little better than rude approximations, adopted from convenience or necessity, depending on no definite principles, and consequently giving results but little entitled to confidence. It appears to me that these notions are attributable, in no small degree, to the manner in which the expressions in question have been spoken of by the writers, Messrs. Morgan and Baily, by whom they were first deduced. Thus, Mr. Morgan says*, "As it is nearly an equal chance that either A or B dies first in any particular year, the first contingency (that of both lives failing in the first year, B last) may be fairly expressed by $\frac{a-b}{a} \times \frac{m-n}{m} \times \frac{1}{2}$;" and Mr. Baily says†, "In all these annual contingencies I have supposed it to be an *equal* chance that A dies before B, whatever may be the difference between the two ages; and though this is not strictly correct, except in those cases where the two lives are equal, yet as such chance is, in the present investigation, confined to *each particular year*, and, when applied practically, is involved with other quantities which very much diminish the error that would otherwise arise from this hypothesis, it would be needless to render the solution more complicated and intricate by aiming at greater exactness. It will however appear from this remark, that the result here obtained is only an *approximation* to the true value, which is more and more correct the further we continue the series." I think it is fairly deducible from these remarks, that neither of the writers was aware that the hypothesis as to the distribution of the yearly deaths (in connexion with which the entire difficulty consists), involved in the assumption of $\frac{1}{2}$ of which they speak, is one of which they freely make use in other investigations,—I mean that the deaths of each year are uniformly distributed over that year, which is the hypothesis in accordance with which the expressions for the mean duration of single and joint lives are usually assigned. Had they been aware of this, it is hardly conceivable that they would have thought it requisite on this occasion to employ the deprecatory and apologetic language above quoted.

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* Doctrine of Assurances, second edition, p. 161, note xvii.

† Doctrine of Life Annuities, p. 109, note.

It seems natural enough, when those by whom the expressions under consideration were devised speak of them in the terms cited, that many of those who come after them, not seeking to go further into the subject than their predecessors, should contentedly imbibe the views suggested, and thus come to look upon the expressions in question with feelings of distrust.

It is the object of the present paper to show, first, that the expressions usually assigned as the measures of the probabilities of survivorship of two lives are *rigorously true* on the hypothesis of a uniform distribution of the deaths of each year; secondly, that this hypothesis is by no means an unreasonable one, and that its results consequently cannot be far from the truth; and thirdly, by the employment of a different and unquestionably more accurate hypothesis, to show what the error of the usual hypothesis in extreme cases really is. I confine myself for simplicity to two lives, but the principles made use of are equally applicable to cases in which the number of lives involved is any whatsoever.

First, the problem to be solved is,—Required the probability that, of two lives now aged x and y respectively, the latter (y) will die in the nth year from the present time, and the former (x) be alive at the moment of (y)'s death. Agreeably to the usual notation, the tabular numbers now alive of the respective ages x and y, are denoted by l_x and l_y; the numbers who out of these enter upon the nth year are denoted by l_{x+n-1} and l_{y+n-1}; and the numbers who complete that year by l_{x+n} and l_{y+n} respectively. And hence the numbers of the two classes who die in the *n*th year fall to be denoted by $l_{x+n-1} - l_{x+n}$, and $l_{y+n-1} - l_{y+n}$ respectively. But for our present purpose it is convenient to employ a somewhat less cumbrous system of notation. Thus I shall denote the numbers of the two classes—

now alive, by	A and B
who enter on the *n*th year, by .	a and b
who survive that year, by . . .	a' and b'
and who die in it, by	d and d'

Now the event in question will happen if (y) die in the *n*th year, and (x) either survive it or die in it after (y). It has thus two ways of happening, and only two; namely, first, by (y) dying in the year and (x) surviving it; and secondly, by both lives failing in the year, (y) first of the two; and the total probability will be the sum of the two partial probabilities in respect of these ways. Now a partial probability is determined by supposing the way of occurrence which gives rise to it as certain to have place, and multiplying the probability derived from that supposition by the probability of the supposition. And this process is to be applied to each way of happening in succession.

Suppose then that the event is to happen in the first of the two ways. It is then certain to happen, the measure of which degree of probability is unity; and the probability of this way of happening being $\frac{a'}{A} \cdot \frac{d'}{B}$, or $\frac{a'd'}{AB}$, this is also, since it is unaffected by multiplication by unity, the partial probability in respect of the first way.

Suppose, again, that the event is to happen in the second way, namely, by both lives failing in the year, (y) first of the two. It is now necessary to make some hypothesis as to the distribution of the deaths of the *n*th year. Let this hypothesis be that the deaths of each class are uniformly distributed over the year. From this hypothesis it follows, that, before the year is entered upon, each of the two lives, (x) and (y), is as likely to die in any one portion of the year as in any other equal portion; for the measure of each such probability will be a fraction of which both the numerator and the denominator are constant. Both lives then being by the supposition certain to fail in the year, and each being, by the mode of distribution, as likely to fail in any one portion of it as in any other equal portion, it undeniably follows, that it is an even chance which of them shall fail the first; that is, in the second way of happening, the probability that either life, as (y), will fail before the other, (x), is $\frac{1}{2}$; and the probability of this way of happening being $\frac{d}{A} \cdot \frac{d'}{B}$, or $\frac{dd'}{AB}$, we have for the partial probability in respect of this way, $\frac{dd'}{AB} \times \frac{1}{2}$, or $\frac{\frac{1}{2}dd'}{AB}$. Hence, adding together the two partial probabilities we have obtained, we have for the total probability required,

$$\frac{a'd' + \frac{1}{2}dd'}{AB}, \text{ or } \frac{d'(a' + \frac{1}{2}d)}{AB}.$$

And if in this we restore the values of A, B, a', d and d', we finally obtain

$$\frac{1}{2}(p_{x.n-1} + p_{x.n})(p_{y.n-1} - p_{y.n}),$$

which is the usual expression*, $p_{x.n-1}$, &c. denoting the probability that (x) will live $n-1$ years, &c.

I do not expect that, although I have been so full, even to diffuseness, in the foregoing demonstration, those who have been accustomed to object to the expression deduced, will have their objection removed by what I have advanced. The objection is taken to the adoption of $\frac{1}{2}$ as the probability that, when both lives fail in the nth year, either of them, whatever may be the difference of their ages, will fail first. Now although I believe the demonstration to be upon that point, and according to the hypothesis assumed as to the distribution of the deaths, absolutely impregnable, I shall, to leave if possible no room for cavil, instead of seeking to strengthen it by further illustration, give another demonstration, to which I think there will be no room to take exception.

Retaining the symbols already employed, suppose that the nth year is divided into m equal parts, m being any whole number; then the probability that (x) will live over t of those parts (t being a whole number less than m), will be

$$\frac{1}{A}\left(a - \frac{td}{m}\right), \text{ or } \frac{1}{mA}(ma - td);$$

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* Jones, p. 151.

for, the deaths being uniformly distributed, those of each mth part of a year will be $\frac{d}{m}$, those of the t parts will be $\frac{td}{m}$, and hence the survivors of a who enter on the year, at the end of the t parts, will be $a-\frac{td}{m}$. Also, the probability that (y) will die in the tth part will be $\frac{d'}{m\text{B}}$, $\frac{d'}{m}$ being the number who, out of (y)'s class, die in each part, on the same hypothesis of uniformity of distribution. Hence, multiplying together these two probabilities, we have

$$\frac{d'}{m^2\text{AB}}(ma-td)$$

for the probability of the compound event, which consists in the failure of (y) in the tth part and the surviving of (x) at the end of that part. If, therefore, in the expression deduced we make t equal to 1, 2, 3 .... m successively, and add the results, the sum will be the probability that (y) will die in some one or other of the m parts into which the year is supposed to be divided, and that (x) will be alive at the end of the same part. Thus, replacing in the meantime by Q the factor $\frac{d'}{m^2\text{AB}}$, we have

$$\begin{array}{ll} t=1, & \text{Q}(ma-d) \\ t=2, & \text{Q}(ma-2d) \\ t=3, & \text{Q}(ma-3d) \\ \cdot\ \cdot & \cdot\ \cdot \\ \cdot\ \cdot & \cdot\ \cdot \\ \cdot\ \cdot & \cdot\ \cdot \\ t=m, & \text{Q}(ma-md). \end{array}$$

Hence, restoring the value of Q, and since $1+2+3\ldots+m = \frac{m(m+1)}{2}$, we have for the probability in question,

$$\frac{d'}{m^2\text{AB}}\left(m^2a-\frac{m(m+1)}{2}d\right), \text{ or } \frac{d'}{\text{AB}}\left(a-\frac{m+1}{2m}d\right).$$

This, as just stated, is the probability that (y) will die in the nth year, and that (x) will survive him by a portion of time not exceeding one-mth part of a year. Now it is obvious, that as this term of probable survivorship is diminished, the compound event whose probability has just been determined will become more and more nearly assimilated to that whose probability we seek, and that without limit; and the requisite change will be made in the formula by supposing m, the number of the parts into which the year is divided, to increase without limit. Hence the limit of $\frac{m+1}{2m}$ being obviously $\frac{m}{2m}$ or $\frac{1}{2}$, we have for the required probability,

$$\frac{d'(a-\frac{1}{2}d)}{AB}.$$

And this is the same as that already found; for, since $a-a'=d$, $\therefore a-\frac{1}{2}d=a'+\frac{1}{2}d$, by transposition.

It is possible that some, unaccustomed to the processes of the higher analysis, may object to the adoption of $\frac{1}{2}$ as the limit of $\frac{m+1}{2m}$, when m is indefinitely increased. Such may satisfy themselves that the error which *seems to them* to exist is inconceivably small, by assuming for m some high number, say 100,000,000 (which by the terms of the demonstration is allowable); then, $\frac{m+1}{2m}$ being *always* equal to $\frac{1}{2}+\frac{1}{2m}$, in this case it becomes $\frac{1}{2}+\frac{1}{200,000,000}$, the employment for which of $\frac{1}{2}$ is more than sufficiently near for all conceivable purposes.

Secondly. It being thus established that the expression usually assigned as the measure of the probability under consideration is rigorously true, on the hypothesis of a uniform distribution of the deaths of each year, I have now to adduce certain considerations tending to show that the adoption of the hypothesis in question cannot be productive of any error of moment.

1. A uniform distribution of the deaths of each year does not imply, as at first sight it might appear to do, a stationary rate of mortality during the periods of uniform distribution. It implies, on the contrary, a constantly increasing rate during those periods; for the probabilities, at the commencement of each succeeding equal portion of a year, of a specified life failing during that portion, form a series of increasing fractions; since the numerators (the numbers who die in each portion) are the same, and the denominators (the numbers who enter upon the successive portions, and out of whom consequently the deaths take place) are in a state of constant diminution. And this is a condition that there can be little doubt ought to be fulfilled by

any hypothesis as to the distribution of the deaths, in behalf of which a claim to be an approximation to the truth can be set up. It is true that, corresponding to variations in the yearly decrements, changes, more or less abrupt, in passing from one year to the next, may take place in the rate of increase spoken of, and that at those points the function may even undergo a sudden diminution. But similar changes must take place at the same points whatever hypothesis may be adopted; although, if this be well chosen, they may be made to assume a gradual instead of an abrupt character. According to De Moivre's hypothesis, the deaths are uniformly distributed over the whole of life; and it is well known how nearly the results of that hypothesis approximate to the truth during what may be considered the most important years of existence.

2. The probability under consideration consists of two portions, of which one is very much greater than the other; a, except in the very last years of existence, always greatly exceeds $\frac{1}{2}d$. Hence, as it is only in the estimation of the smaller portion that the necessity for taking account of the distribution of the deaths arises, it follows that a small error arising from the employment of any particular hypothesis will have a proportionately smaller effect on the total probability.

Thirdly. I am now, with a view to determine the amount in extreme cases of the error committed by the employment of the hypothesis of uniform distribution, to present a solution of the problem on another, and undoubtedly a more correct hypothesis. For facility in the treatment of the hypothesis in view, however, and indeed in that of any hypothesis which shall aim at greater accuracy than the one heretofore employed, it is desirable to be in possession of methods of greater power than that which has hitherto sufficed. The method I shall employ is that of the Calculus of Finite Differences, on the application of which to the summation of series a few words may be necessary.

Let $\phi(x)$ denote any function of x; and suppose that in this function x is changed into $x+1$. It will now be denoted by $\phi(x+1)$. The original function will consequently have undergone a change, received an accession of value (which may be either positive or negative according to circumstances) amounting to $\phi(x+1)-\phi(x)$, which is called the difference of $\phi(x)$, and is denoted by $\Delta\phi(x)$. It is the province of the Direct Calculus of

Differences to determine $\Delta\phi(x)$ for every form of $\phi(x)$. Again, consider $\phi(x)$ as a difference arising from a change of the kind just described in another function. The determination of this other function, which is called the primary function, or the finite integral of $\phi(x)$, and is denoted by $\Sigma\phi(x)$, is the province of the Inverse Calculus of Differences, and the process is called finite integration.

Now let S_x and S_{x+1} denote the sum of x and of $x+1$ terms of the same series respectively. It is obvious that the difference between these is the $(x+1)$th term, say u_{x+1}; that is, $S_{x+1}-S_x$, or $\Delta S_x=u_{x+1}$. Hence $\Sigma u_{x+1}=S_x$. Consequently, by the integration of the $(x+1)$th term, we obtain the sum of the series, and are thereby enabled to dispense with the cumbrous process that without this aid would be unavoidable.

As introductory to the application of this process to the new hypothesis, I shall here repeat, upon the principles just enunciated, the solution last given of the problem under consideration, upon the old hypothesis. Let α and β denote the probabilities that (x) and (y) will enter upon the nth year respectively. Then, upon the hypothesis of uniform distribution, the probabilities that they will be alive at the end of any portion of that year, as $\frac{t}{m}$, will be denoted by $\alpha+\frac{t}{m}\Delta\alpha$, and $\beta+\frac{t}{m}\Delta\beta$ respectively. Hence also, since $\beta+\frac{t-1}{m}\Delta\beta$ will be the probability that (y) will be alive at the end of the time $\frac{t-1}{m}$, the probability that he will die between $\frac{t-1}{m}$ and $\frac{t}{m}$, that is in the tth in order of the m parts into which the year is divided, will be

$$\beta+\frac{t-1}{m}\Delta\beta-\beta-\frac{t}{m}\Delta\beta^{*}=-\frac{1}{m}\Delta\beta.$$

Consequently, the probability of (y) dying in the tth part and (x) surviving it, will be

$$-\frac{1}{m}\Delta\beta\left(\alpha+\frac{t}{m}\Delta\alpha\right), \text{ or } -\frac{1}{m^2}\Delta\beta(m\alpha+t\Delta\alpha).$$

Now changing t into $t+1$, integrating, and in the integral replacing t by m, we have for the probability in respect of the whole year,

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[*] Milne, p. 52. Mr. Milne's demonstration of the property here made use of has reference to entire years only; but the property is equally true of any portion of time.

$$-\frac{1}{m^2}\Delta\beta\left(m^2\alpha+\frac{m(m+1)}{2}\Delta\alpha\right),\ \text{ or }\ -\Delta\beta\left(\alpha+\frac{m+1}{2m}\Delta\alpha\right).$$

Finally, taking the limit of $\frac{m+1}{2m}$, we have for the probability required,

$$-\Delta\beta(\alpha+\frac{1}{2}\Delta\alpha);$$

and this is the same as the expression already obtained, $\Delta\alpha$ and $\Delta\beta$ being essentially negative.

Now what after all is the defect of the hypothesis of uniform distribution, on which we have hitherto proceeded? Undoubtedly this: the mortality table shows a series of decrements in most cases varying from year to year, either by increase or decrease; and these yearly variations must unquestionably be held as indicating corresponding variations in the distribution of the mortality of the individual years, amongst the component parts of those years. Thus if, for example, the mortality of the $(n+1)$th year be greater than that of the nth, then the mortality during any portion of the nth year towards its close ought to be greater than the mortality during any equal portion towards its commencement, and *vice versâ*. It is here then that the hypothesis in question is defective, inasmuch as it assigns, in all cases, to equal portions of each year the same number of deaths. Thus, if the deaths of the nth year are 100, the uniform hypothesis would assign 25 of those to each quarter of that year, irrespective of the number of the deaths of the $(n+1)$th and following years; whereas if the deaths of the $(n+1)$th year are 132 or 68, the quarterly distribution of the nth year would undoubtedly be much better represented, in the one case by 22, 24, 26, 28, and in the other by 28, 26, 24, 22; because while those series give the deaths of the nth year correctly, their continuation for four terms further would give the deaths of the $(n+1)$th year correctly also.

Hence in assigning the present probability of a specified life being in existence at the end of a fractional part of the nth year (or, which is the same thing, the number surviving out of a given number now alive), it is necessary to take account of the deaths of the $(n+1)$th year, as well as those of the nth. Indeed, perfect accuracy would prescribe the taking account of the deaths of every year after the nth; but as the state of the decrement of the $(n+1)$th year exercises an incomparably greater influence on the distribution of the nth year than the state of the decrements in all the following years together, and as we should find that influence to be, even in extreme cases, comparatively small, it would be needlessly complicating our problem to go beyond the $(n+1)$th year. In accordance with this principle, the present probabilities of the specified life being alive at the end of $n-1$, n, and $n+1$ years, are treated as equidistant terms of a series, and the common formula of differences then enables us at once to assign any term intermediate to $n-1$ and n.

Let, as before,

$\alpha=$ the probability that (x) will live over $n-1$ years,
$\beta=$ the probability that (y) will live over $n-1$ years,

then, on the present hypothesis, the probability that (x) will be alive at the end of the time $\frac{t}{m}$ of the nth year (m being any whole number, and t not greater than m) will be,

$$\alpha+\frac{t}{m}\Delta\alpha+\frac{t(t-m)}{2m^2}\Delta^2\alpha; \quad . \; . \; . \; . \; . \; . \quad (1.)$$

and the like probability for (y) will be

$$\beta+\frac{t}{m}\Delta\beta+\frac{t(t-m)}{2m^2}\Delta^2\beta. \quad . \; . \; . \; . \; . \; . \quad (2.)$$

Also the probability that (y) will be alive at the end of the time $\frac{t-1}{m}$ will be

$$\beta+\frac{t-1}{m}\Delta\beta+\frac{(t-1)(t-m-1)}{2m^2}\Delta^2\beta. \quad . \; . \; . \quad (3.)$$

Consequently the difference of (2.) and (3.), or

$$-\frac{1}{m}\left(\Delta\beta+\frac{2t-m-1}{2m}\Delta^2\beta\right), \quad . \; . \; . \; . \quad (4.)$$

will be the probability that (y) will die between $\frac{t-1}{m}$ and $\frac{t}{m}$. Hence, multiplying together (1.) and (4.), we have for the probability that (y) will die between $\frac{t-1}{m}$ and $\frac{t}{m}$, and that (x) will be alive at the end of $\frac{t}{m}$,

$$\begin{aligned}-\frac{1}{m}\Big\{\alpha\,.\,\Delta\beta+\frac{t}{m}\Delta\alpha\,.\,\Delta\beta+\frac{t(t-m)}{2m^2}\Delta\beta\,.\,\Delta^2\alpha+\frac{2t-m-1}{2m}\alpha\,.\,\Delta^2\beta \\ +\frac{t(2t-m-1)}{2m^2}\Delta\alpha\,.\,\Delta^2\beta+\frac{t(t-m)(2t-m-1)}{4m^3}\Delta^2\alpha\,.\,\Delta^2\beta\Big\}.\end{aligned}$$

Writing in this expression $t+1$ for t, and integrating between the limits 0 and m, we have for the foregoing probability in respect of the whole of the nth year,

$$-\left\{\alpha.\Delta\beta+\frac{m+1}{2m}\Delta\alpha.\Delta\beta-\frac{m^2-1}{12m^2}\left(\Delta^2\alpha.\Delta\beta-\Delta\alpha.\Delta^2\beta-\frac{1}{2m}\Delta^2\alpha.\Delta^2\beta\right)\right\}.$$

Finally, increasing m without limit, we obtain for the probability required,

$$-\alpha.\Delta\beta-\frac{1}{2}\Delta\alpha.\Delta\beta-\frac{1}{12}(\Delta\alpha.\Delta^2\beta-\Delta^2\alpha.\Delta\beta).$$

I shall return to this expression. In the meantime I remark, that, much as the labour of the summation by which it is arrived at is reduced by the employment of the processes of the Calculus of Differences, it is still far from being inconsiderable. It will therefore perhaps be deemed not out of place if I briefly direct attention to another method, by the aid of which not only the hypothesis now under consideration, but also any other which it may be desired to examine, may be treated with great facility. The method now in view, which involves an elementary application of the Integral Calculus, is indicated by Professor De Morgan in the Third Appendix to his Essay on Probabilities, and there applied to the hypothesis at present under consideration. I need do little more than supply the necessary amplification*.

Let then ϕt and ψt (ϕ and ψ being as usual functional symbols) denote the probabilities that (x) and (y), now alive, shall be alive at the end of t years, t being whole or fractional. Let t become $t-\Delta t$; then, expanding by Taylor's theorem, the probability that (y) will be alive at the end of $t-\Delta t$, will be,

$$\psi t-\psi' t.\Delta t+\psi'' t\frac{(\Delta t)^2}{2}-\ldots.$$

where $\psi' t$, $\psi'' t$, &c. are the successive differential coefficients of ψt. But when Δt is diminished towards its limit dt, this becomes

$$\psi t-\psi' t.dt,$$

since $(\Delta t)^2$, &c. are infinitely small in comparison of Δt; and it then denotes the probability that (y) will be alive at the end of $t-dt$. Hence the probability that (y) will die between $t-dt$ and t, will be

$$(\psi t-\psi' t.dt)-\psi t=-\psi' t.dt.$$

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* I have however slightly modified the process to bring it into accordance with the foregoing investigations.

Consequently we have for the probability that (y) will die between $t-dt$ and t, and (x) be alive at the end of t,

$$-\phi t \,.\, \psi' t \,.\, dt.$$

And the integral of this expression, which is written thus,

$$-\int \phi t \,.\, \psi' t \,.\, dt,$$

between the limits $t=n-1$ and $t=n$, will be the probability that (x) will commence to survive (y) somewhere in the nth year.

This remarkable expression is quite general. It implies that, be the functions employed to denote the probabilities of (x) and (y) surviving the time t what they may, the integral of the product

$$-\phi t \,.\, \psi' t \,.\, dt,$$

between the specified limits, will always denote the probability of a survivorship of (x) commencing in the nth year.

Apply this expression to our present hypothesis, that is, let

$$\phi t = \alpha + t\Delta\alpha + \frac{t(t-1)}{2}\Delta^2\alpha,$$

and

$$\psi t = \beta + t\Delta\beta + \frac{t(t-1)}{2}\Delta^2\beta,$$

t denoting not, as formerly, a number, but a portion of time measured from the commencement of the nth year. Differentiating ψt, we have

$$\frac{d(\psi t)}{dt}, \text{ or } \psi' t = \Delta\beta + \left(t-\frac{1}{2}\right)\Delta^2\beta.$$

Hence, multiplying by ϕt, we have

$$\phi t \,.\, \psi' t = \alpha \,.\, \Delta\beta + t\Delta\alpha \,.\, \Delta\beta + \frac{t(t-1)}{2}\Delta^2\alpha \,.\, \Delta\beta + \left(t-\frac{1}{2}\right)\alpha \,.\, \Delta^2\beta$$
$$+ t\left(t-\frac{1}{2}\right)\Delta\alpha \,.\, \Delta^2\beta + \frac{t\left(t-\frac{1}{2}\right)(t-1)}{2}\Delta^2\alpha \,.\, \Delta^2\beta.$$

Multiplying by dt, changing signs, and integrating between the limits $t=0$ and $t=1$, we have, finally, for the required probability,

$$-\int_0^1 \phi t \,.\, \psi' t \,.\, dt = -\alpha \,.\, \Delta\beta - \frac{1}{2}\Delta\alpha \,.\, \Delta\beta - \frac{1}{12}(\Delta\alpha \,.\, \Delta^2\beta - \Delta^2\alpha \,.\, \Delta\beta),$$

the fourth and sixth terms vanishing between the specified limits. And this is the expression already found, but it is here arrived at by a very much smaller expenditure of labour.

Now observe of this expression, that the first two terms of it are those that express the entire probability on our first hypothesis of uniformity of distribution. The remaining term is consequently the correction arising from the adoption of the new hypothesis.* To put the expression in a form for use, substitute in it,

for α, $p_{x.n-1}$,

$\Delta\alpha$, $p_{x.n}-p_{x.n-1}$,

$\Delta^2\alpha$, $(p_{x.n+1}-p_{x.n})-(p_{x.n}-p_{x.n-1})=p_{x.n+1}-2p_{x.n}+p_{x.n-1}$;

and similarly for β, $\Delta\beta$ and $\Delta^2\beta$; and it then becomes

$$\frac{(p_{x.n-1}+p_{x.n})(p_{y.n-1}-p_{y.n})}{2}$$

$$-\frac{(p_{x.n-1}-p_{x.n})(p_{y.n}-p_{y.n+1})-(p_{x.n}-p_{x.n+1})(p_{y.n-1}-p_{y.n})}{12},$$

or

$$\frac{(l_{x+n-1}+l_{x+n})(l_{y+n-1}-l_{y+n})}{2l_{x.y}}$$

$$-\frac{(l_{x+n-1}-l_{x+n})(l_{y+n}-l_{y+n+1})-(l_{x+n}-l_{x+n+1})(l_{y+n-1}-l_{y+n})}{12l_{x.y}}.$$

As before, the first part of this expression is that which is usually given, and the second is the correction. This last, it is easy to see, must always be very small; for it consists of the difference of two products, both of which are small and very nearly equal, each factor in the one product having a corresponding and nearly equal factor in the other. The correction, for the same pair of lives, will be in some years positive, and in others negative; and it is worth while to remark, that it entirely vanishes, becomes rigorously zero, if for both lives the deaths of the nth and $(n+1)$th years are equal; that is, if the decrements of those years for both lives are stationary.

It is a practicable task now, from the second form of the expression above given, to determine the value of the correction for any year of the future existence of two lives, according to any table of mortality. I do not stop to give examples for single years, but proceed to deduce from the expression the corresponding form for the whole life. We have then for the total probability that of the two lives (x) and (y), (y) will die first,

$$\frac{1}{2}+\frac{N_{x.y-1}-N_{x-1.y}}{2D_{x.y}}$$

$$+\frac{(N_{x-1.y+1}+N_{x.y-1}+N_{x+1.y})-(N_{x-1.y}+N_{x.y+1}+N_{x+1.y-1})}{12D_{x.y}}.$$

This formula may be applied by means of Mr. Jones's Table XL.; and the following are a few results of its application. Of the two values corresponding to each pair of ages, the first, which is derived from the first two terms of the foregoing expression, is the usual approximation; and the second, which is derived from the remaining term, is the correction. In all the examples the correction is additive:—

Probability that (y) will die before (x).

x.	y.					
	30.	40.	50.	60.	70.	80.
20.	·625094 ·000050	·730322 ·000065	·811215 ·000043	·880591 ·000044	·928970 ·000058	·959724 ·000091
30.	…………	·633800 ·000059	·749185 ·000076	·842126 ·000062	·903450 ·000061	·943960 ·000116
40.	…………	…………	·650231 ·000072	·787778 ·000115	·873990 ·000097	·925482 ·000142
50.	…………	…………	…………	·684674 ·000125	·822997 ·000208	·907104 ·000296
60.	…………	…………	…………	…………	·677755 ·000157	·809907 ·000384
70.	…………	…………	…………	…………	…………	·679920 ·000559

It is thus shown how small the correction is, within the limits which usually occur in practice; and when it is considered how laborious the application of the more correct formula is, I do not anticipate that it will often be thought necessary to have recourse to it. It is however now available for all who choose to make use of it.

The foregoing investigation gives, of course, an addition to the usual expression for the value of a survivorship assurance. The numerical value of the correction in this case will be considerably less than in the case that has been considered, in consequence of the entrance here of the element of interest. I subjoin the resulting formulæ, without examples:—

Value of an Assurance of £1 on (y) against (x).

$$y > x$$

$$\frac{v(\mathrm{N}_{x-1\,.\,y-1}+\mathrm{N}_{x\,.\,y-1})-(\mathrm{N}_{x-1\,.\,y}+\mathrm{N}_{x\,.\,y})}{2\mathrm{D}_{x\,.\,y}}$$

$$-\frac{(\mathrm{N}_{x-1\,.\,y}-\mathrm{N}_{x+1\,.\,y})-v(\mathrm{N}_{x\,.\,y-1}-\mathrm{N}_{x+1\,.\,y-1})-(1+r)\,(\mathrm{N}_{x-1\,.\,y+1}-\mathrm{N}_{x\,.\,y+1}}{12\mathrm{D}_{x\,.\,y}}$$

$$y < x$$

$$\frac{v(\mathrm{N}_{x-1\,.\,y-1}-\mathrm{N}_{x-1\,.\,y})+(\mathrm{N}_{x\,.\,y-1}-\mathrm{N}_{x\,.\,y})}{2\mathrm{D}_{x\,.\,y}}$$

$$-\frac{v(\mathrm{N}_{x-1\,.\,y}-\mathrm{N}_{x-1\,.\,y+1})+(1+r)(\mathrm{N}_{x+1\,.\,y-1}-\mathrm{N}_{x+1\,.\,y})-(\mathrm{N}_{x\,.\,y-1}-\mathrm{N}_{x\,.\,y+1})}{12\mathrm{D}_{x\,.\,y}}$$

I conceive that I have now established the positions with which I set out. I have *proved* that the expression usually assigned as the probability of one life surviving another is *rigorously true* on the hypothesis of a uniform distribution of the deaths of each year; and I have shown, by comparison of certain results of that hypothesis with the corresponding results of another, and unquestionably a more correct hypothesis, that the error arising from the employment of the first-mentioned hypothesis is so small as seldom to require to be taken into account. It will therefore now be for such of those who have followed me thus far, as demur to my conclusions, to show the fallacy of my demonstrations and the inconclusive character of my reasonings. Till this be done, I submit that I am warranted in viewing the questions in debate as set at rest.

London, 18th November 1850.

On the Contrivances required to render contingent Reversionary Interests Marketable Securities.

[Continued from No. 2.]

SINCE the loss occasioned by the reassurance varies with the mode of operation, it becomes interesting to inquire in which way the loss may be made the least possible. B is in the situation of a merchant who possesses a commodity not quite in a marketable state, and who seeks to bring it into a condition for sale at the least expense. A little reflection will show that the longer the borrowing is deferred, the less loss will B sustain, on account of the difference at interest; hence we must seek to diminish as much as possible the early payments, and to throw the weight of the premiums to a distant time—a scale of ascending premiums is thus the more advantageous for B. Now the most rapid rate of ascent which an assurance office can allow, is such that each premium may pay for the succeeding year's risk; no office, indeed, will go thus far*; yet it is worth while to examine this case, since it exhibits the minimum loss at which B can procure a reassurance.

CASE 4.—Let the assurance of £1, payable at the death of B,

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* Some will.—ED. A. M.